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Testing inflation with the cosmic microwave background

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Cosmic microwave background (CMB) anisotropy may result from both scalar and tensor perturbations. For a sufficiently narrow range of angular scales, CMB perturbations can be characterized by four parameters. Results from the Cosmic Background Explorer fix one combination of the parameters, reducing the parameters to three. If CMB perturbations are from inflation, there is an additional relation, reducing the parameters to two. An appropriate combination of a medium-angle and a small-angle CMB observation can test the inflation hypothesis because inflation cannot explain a high signal in one experiment and a low signal in the other.

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Guth pointed out that a rapid expansion early in the history of our Universe could solve several problems of the standard cosmology, the most notable being the horizon and flatness problems [1]. It was realized shortly after the initial proposal that inflation does more than provide a smooth Universe: It also generates small perturbations on the smooth background [2]. These small perturbations may well be the ones that grow via gravitational instability into the diversity of structures we see today in our Universe.

Most successful models of inflation involve the dynamics of a weakly coupled scalar field that slowly evolves (rolls) under the influence of some scalar potential. This scalar field is called the *inflaton*. While the inflaton evolves during inflation, perturbations in the energy density arise as the result of quantum mechanical fluctuations of the inflaton field in de Sitter space. These perturbations in the energy density correspond to *scalar* metric fluctuations. In addition to the scalar metric fluctuations there should be *tensor* perturbations caused by de Sitter space fluctuations of the metric tensor [3].

In principle, one must specify the amplitude of the tensor and scalar perturbations on all length scales, each of which enters the horizon at different cosmic times. However, if the inflaton evolves slowly during inflation (as it must in order that its potential energy dominate the energy density) then the resulting scalar perturbations should approximately have the Harrison–Zel'dovich spectrum; the Fourier transform of their variance is a simple power law with exponent $n_S = 1$. In the same slow-roll limit the Fourier transform of the variance of the tensor perturbations is also a power law, with exponent $n_T = 0$.

Of course the fact that the scalar field must evolve during inflation means that the resulting spectra won't have exactly the above simple form. However since the field must evolve slowly, over a limited range of length scales (such as the length scales probed by CMB experiments) it should be possible to describe the scalar and tensor perturbations as power laws, with exponents not too different from the Harrison–Zel'dovich values. Therefore we will assume that the spectra can be described by four parameters: the amplitudes and spectral indices of the scalar and tensor spectra.

The purpose of this Letter is to show that inflation is testable even though there are four parameters which describe the scalar and tensor spectra. In particular we argue that a good test of inflation can be constructed by combining information from three sets of cosmic microwave background anisotropy experiments: COBE [4] at large angular scales; one at medium angular scales, and a third at small angular scales. At first glance this seems a hollow claim, for there are four free parameters in the scalar and tensor spectra, and with four free parameters one should easily be able to fit three experimental results. How can just three experiments test inflation?

Fortunately, inflation does make a generic prediction; namely a relationship between the shape of the tensor spectrum and its amplitude. Thus, there are only three free parameters if the perturbations arise from inflation. Suppose we fix one of these parameters with the COBE result. As we vary the other two, we can indeed significantly change the expected signal in both the small and the medium angle experiments. However, we find that *the signal in the small-angle experiment scales in a predictable way as a function of the signal in the medium-angle experiment.* So once the signal in one of these is fixed, the signal in the other is unambiguously determined. We can think of this abstractly as a mapping of the two-dimensional space of the two remaining free parameters after COBE normalization onto the two-dimensional space of signals in the medium- and small-angle experiments. If the mapping were onto, so that any point in the signal space could be reached by a point in the parameter space, then inflation could never be disproved by this set of experiments. Our claim is that the two dimensional parameter space is mapped into a single one-dimensional line in signal space. A combination of experimental results not falling on this line cannot result from inflation, so there is a whole range of experimental results which can disprove inflation.

Before discussing more precisely what we mean by "medium-" and "small-" angle experiments, we must introduce some notation. It has become standard to decompose the angular correlation function into a sum of Legendre polynomials:

$$C(\theta) \equiv \langle \delta T(\theta) \delta T(0) \rangle = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C_l P_l(\cos\theta). \tag{1}$$

Recall that in this expansion, small l corresponds to large angles while large l corresponds to small angles. Experiments do not directly measure $C(\theta)$, but rather measure some convolution of the C_l's with a window function [5], W_l, determined by the particular experiment. Thus, the predicted variance in a given experiment is defined as

$$\langle \delta T^2_{\mathrm{exp}} \rangle = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C_l W_l. \tag{2}$$

To determine the predictions of a given theory, one must calculate the set of C_l's it predicts [6]. Let's briefly review the steps involved in such a calculation: (i) perturb the Einstein and Boltzmann equations about the standard zero-order solutions (the Robertson-Walker metric with homogeneous and isotropic distributions of photons, neutrinos, ordinary matter, and dark matter); (ii) Fourier transform these equations to express the perturbations Δ in terms of wavenumber k, time t, and in the case of photons and neutrinos, the angle

θ between the wavenumber and momentum; (iii) Expand the perturbations to the photons and neutrinos in terms of Legendre polynomials so that the angular dependence, $\Delta(\theta)$, is replaced by the coefficients, Δ_l; (iv) Evolve these perturbed quantities starting from initial conditions deep in the radiation era: $(\delta\rho/\rho)_S(k, t_{\rm init}) \propto k^{n_S/2}; (\delta\rho/\rho)_T(k, t_{\rm init}) \propto k^{(n_T-3)/2}$ where $n_S = 1; n_T = 0$ for the Harrison–Zel'dovich spectra; (v) Determine the C_l's due to both scalar and tensor modes today by integrating $C_{l,(S,T)} \propto \int d^3k|\Delta_{l,(S,T)}(t_0)|^2$; (vi) Add the two contributions [7]: $C_l = C_{l,S} + C_{l,T}$. The proportional signs in steps (iv) and (v) are an indication that we do not know the normalization of either mode. We can fix one such parameter, say $C_{2,S}$, by using the COBE result [8]. The three remaining unknowns are $R \equiv C_{2,T}/C_{2,S}$; n_S, and n_T.

Now that we have defined the relevant parameters we can discuss the prediction of inflation. The tensor-to-scalar energy density ratio [9], r, as well as the spectral indices can be expressed in terms of the derivatives of the expansion rate, H, during inflation. In the limit that H is constant during inflation, $[r,\ n_T,\ n_S] = [0, 0, 1]$. However in slow-roll inflation H changes in time. Using the value of the inflaton field as the time variable $[\phi = \phi(t)]$, then one can calculate $[r,\ n_T,\ n_S]$ as a function of $\epsilon \equiv 2[H'(\phi)/H(\phi)]/\kappa^2$, $\eta \equiv 2[H''(\phi)/H'(\phi)]/\kappa^2$, and $\xi \equiv 2[H'''(\phi)/H'(\phi)]/\kappa^2$. Here $\kappa^2 \equiv 8\pi/m^2_{\rm Planck}$ and prime denotes $d/d\phi$. If $(\epsilon,\ \eta,\ \xi)$ are less than unity, then to first order in these parameters [10]

$$[r,\ n_T,\ 1 - n_S] = \left[\frac{25}{4}2\epsilon,\ -2\epsilon,\ 2\epsilon\left(2 - \frac{\eta}{\epsilon}\right)\right] \tag{3}$$

Therefore, to first order $r = -6.25 n_T$. A scalar-to-tensor ratio of $r = -6.25 n_T$ results in $R \equiv C_{2,T}/C_{2,S} = -7n_T$.

The set of C_l's predicted by inflation is therefore dependent on two free parameters: $C_l = C_l(n_S, R = -7n_T)$. Fig. 1 shows the C_l's for several different values of these two parameters. The solid curve is for standard inflation $[(R,\ n_S) = (0,\ 1)]$. As n_S increases the signal increases, the effect being greatest on the smallest scales. So the dashed curve, which is for $(R,\ n_S) = (0,\ 1.25)$, is higher than the standard one. As R increases, the signal at small angular scales [large l] goes down: $(R,\ n_S) = (2, 1)$ produces the dotted curve in Fig. 1. The point is that the COBE signal [which comes from $l < 20$] is partly due to tensor modes in this case, thereby reducing the amplitude of the scalar component. The tensor contribution drops off after $l = 100$ (physically this is because gravity waves redshift once they enter the horizon; $l = 100$ is roughly the scale of the horizon at decoupling). So once $l > 150$, all that is left is the reduced scalar contribution. Note however that the signal in the medium angle range $l \sim 50$ also decreases. This is a consequence of the inflationary

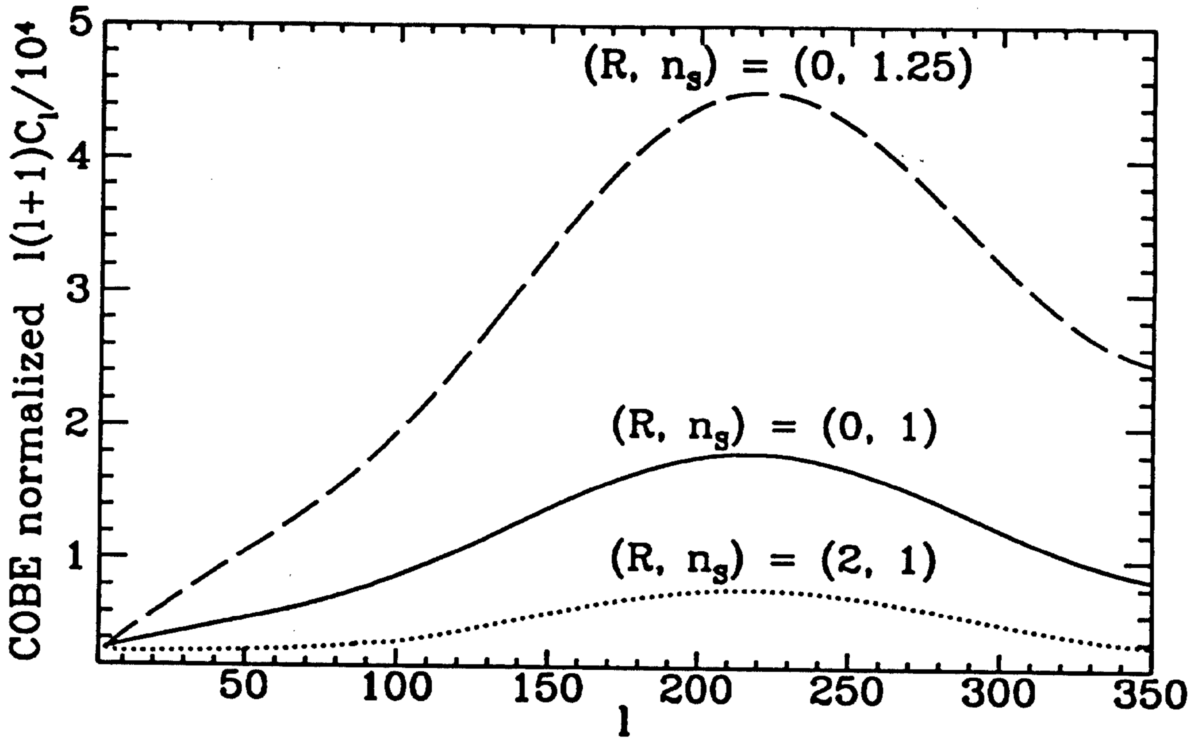

Fig. 1: C_l's predicted for three different inflation models. C_l's are in units of $(\mu K)^2$.

prediction $n_T = -R/7$. Since $R = 2$ in this case, the tensor spectrum is tilted so as to fall off rapidly with increasing l. We can thus imagine that while alternative models of inflation may change the signal in either a small or a medium angle experiment, it will be very hard to change the signals in opposite directions. This observation is the basis for the test we will shortly describe.

Before getting into the details of the test, we should spell out our assumptions. The C_l's in Fig. 1 were generated assuming cold dark matter; zero cosmological constant; standard ionization history; Hubble constant today $H_0 = 50$ km sec^{-1} Mpc^{-1}; and the fraction of critical density in baryons today, $\Omega_B = 0.05$. Varying any of these parameters can lead to significant changes in the C_l's [11]. Our philosophy is that these parameters, while not particularly well-determined today, are very likely to be determined in the future *by experiments other than microwave anisotropy experiments* [12]. Therefore, we feel it is unlikely that our lack of knowledge about these parameters will be the stumbling block keeping us from testing inflation.

Now for a method to test inflation. Imagine two anisotropy experiments, one with a medium-angle filter, $W_l^{(1)} = 1$ for $30 < l < 90$ and zero otherwise; the other with a small-angle filter, $W_l^{(2)} = 1$ for $130 < l < 300$ and zero otherwise. Then the predicted signal in each experiment is

$$\begin{aligned} \delta T^{(1)}(n_S, n_T, R) &= \left[\sum_{l=30}^{90} \frac{2l+1}{4\pi} C_l(n_S, n_T, R)\right]^{1/2} \\ \delta T^{(2)}(n_S, n_T, R) &= \left[\sum_{l=130}^{300} \frac{2l+1}{4\pi} C_l(n_S, n_T, R)\right]^{1/2} . \end{aligned} \tag{4}$$

The first experiment would sample both scalar and tensor modes, while the second would sample only scalar modes. We have explicitly indicated that the signal in these experiments depends on the values of (n_S, n_T, R). The set of parameters allowed by inflation can now be mapped onto these two signals. Fig. 2 shows such a mapping for $-0.5 < n_S < 1.5$; $R < 3.5; n_T = -R/7$. A larger range would not be consistent with our use of equation 3, which is true only to first order in the slow-roll parameters, ϵ and η. The important point is that this whole region of parameter space allowed by inflation is mapped onto a very narrow region, almost a line, in signal space [13]. There is some scatter off this line, particularly if the signals are low. However, this low signal regime comes from $n_S < 0.5$; such small values of n_S, as we discuss later, are highly unlikely given other data.

Inflation therefore predicts small deviations from this "line" in signal space. We can quantify this further by defining a linear combination of the signals such that one of the new variables is the distance from the locus of inflation predictions. A good measure of this distance is

$$\begin{aligned} D \equiv\ & \alpha \left[\frac{\delta T^{(1)}}{\delta T^{(1)}(n_S = 1; R = 0)} - 1\right] \\ & -\beta \left[\frac{\delta T^{(2)}}{\delta T^{(2)}(n_S = 1; R = 0)} - 1\right] \end{aligned} \tag{5}$$

where

$$\begin{aligned} \alpha &\equiv \frac{1}{\sqrt{1+\gamma^2}} \quad ; \quad \beta \equiv \frac{1}{\sqrt{1+\gamma^{-2}}} \\ \gamma &\equiv \frac{1 - \delta T^{(1)}(n_S = .5, R = 0)/\delta T^{(1)}(n_S = 1, R = 0)}{1 - \delta T^{(2)}(n_S = .5, R = 0)/\delta T^{(1)}(n_S = 1, R = 0)} . \end{aligned}$$

Note that α, β, and γ can be calculated for any pair of filter functions. For the simple square ones [14] we have chosen, $\alpha = 0.77$ and $\beta = 0.63$. So D is roughly the difference between

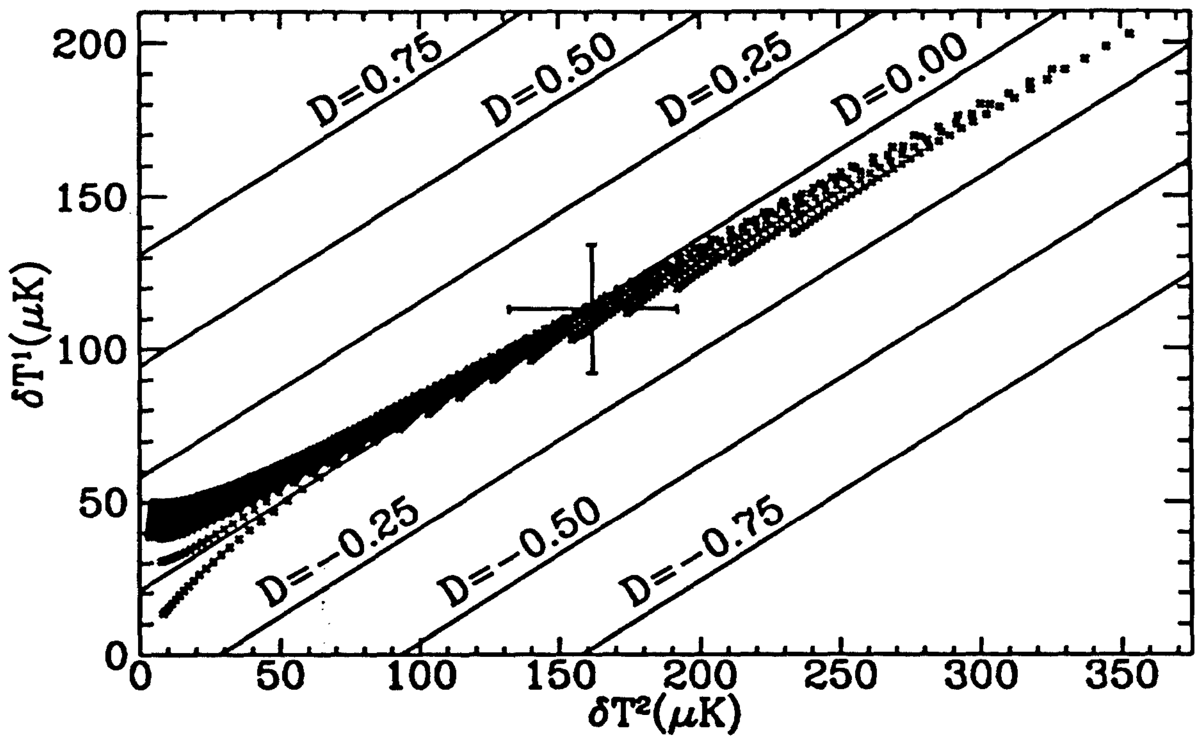

Fig. 2: A mapping from the plane of inflation predictions into the signal plane of the medium ($\delta T^{(1)}$) and small ($\delta T^{(2)}$) angle experiment plane. Also shown are contours of constant D, defined in Eq. 5, and the error bars [centered around the ($n = 1, R = 0$) prediction] for experiments with 50 pixels and unity signal/noise ratio.

the signals in the small and medium angle experiments, each of which is normalized by its standard value at $n_S = 1$. The fact that inflation predicts $D \simeq 0$ means that *inflation cannot explain a high signal in one of these experiments and a low one in the other.*

Fig. 3 shows D as a function of n_S and $R = -7n_T$. As mentioned above, D does begin to deviate from zero, but only in "non-physical" regions in the parameter space. A way to quantify this is to note that σ_8, the *rms* mass fluctuation in spheres of radius $8h^{-1}$Mpc, must certainly be greater [15] than about 1/3, while the light region in Fig. 3 has $\sigma_8 < 1/3$. [On the color version, colors redder than yellow in Fig. 3 have $\sigma_8 < 1/3$.] So the value of D in these regions is irrelevant. In the "allowed" range, D is always less than 0.06. When $n_S > 1$, D can become negative, as small as -0.1; however, such large values of n_S are also thought to be unlikely because they would produce too much structure on scales of about 1 Mpc.

We have defined a new parameter D [Eq. (5)] which combines information from a small

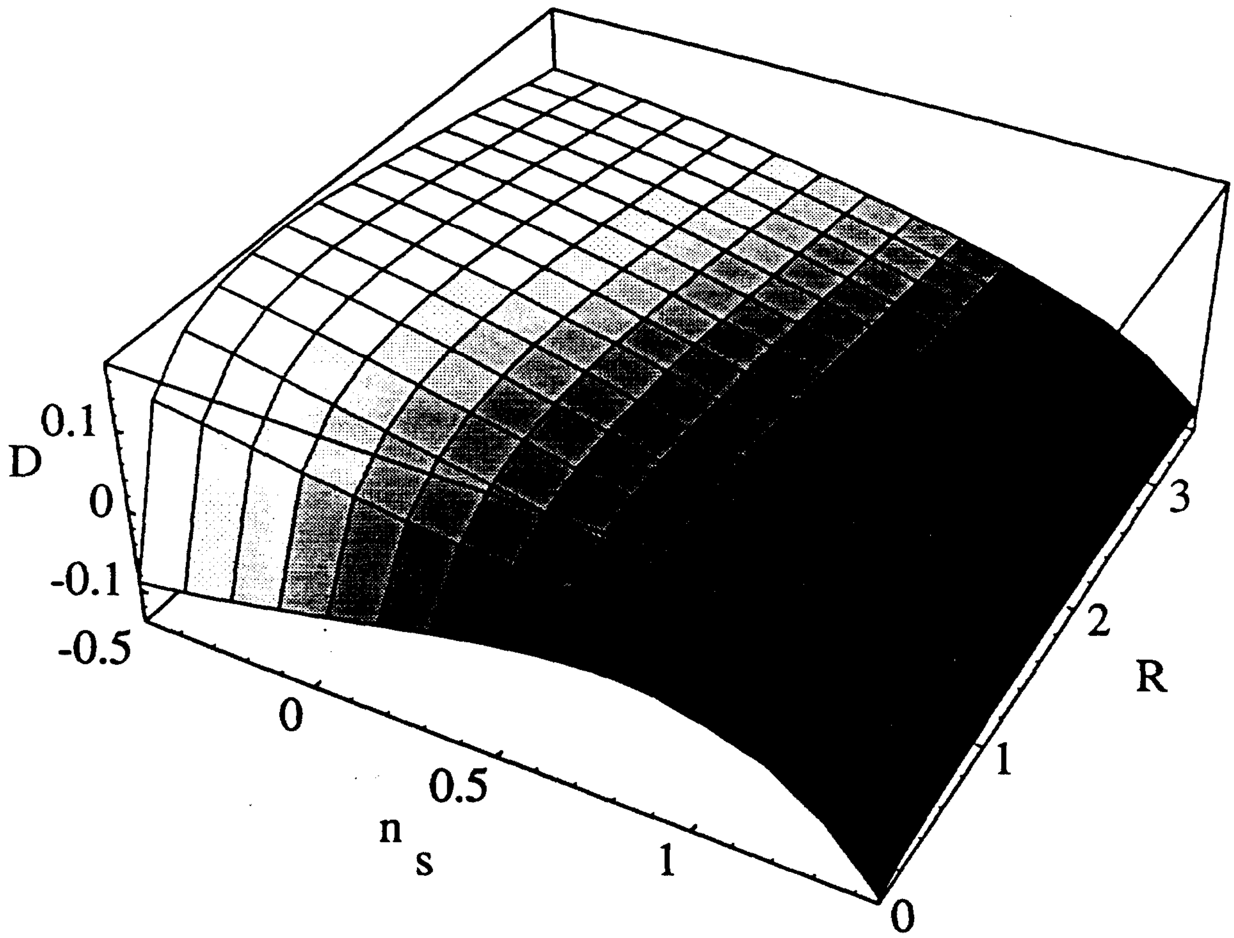

Fig. 3: The parameter D of Eq. (5) as a function of n_S and R.

and a medium angle experiment. Many experiments currently on-line are probing the angular regimes necessary to evaluate D. Viable models of inflation predict $D < 0.06$. It remains to be seen what values of D are predicted by other cosmological theories, such as those with non-Gaussian seeded perturbations.

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[12] The ionization history is the one area where it is not clear what experiments can be done to test the standard assumption. Particularly thorny is the issue of late reionization.

[13] The low scatter in Figure 2 also means that a simple model of inflation with no tensor modes [see e.g. Ref. [8]] is an effective way of probing all inflationary models. That is, by varying n alone, we can get any set of signals that might be generated by including tensor modes as well.

[14] We have also tried Gaussian filter functions centered at $l = 60$ and $l = 200$. As long as neither extends much over the $l = 100$ boundary, α and β have the same numerical values given in the text.

[15] In Fig. 3, σ_8 has been determined assuming cold dark matter; any other form of dark matter leads to even smaller values of σ_8.